Time-dependent rotating stratified shear flow: Exact solution and stability analysis

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A solution of the Euler equations with Boussinesq approximation is derived by considering unbounded flows subjected to spatially uniform density stratification and shear rate that are time dependent $[S(t)=\partial U_3/\partial x_2]$. In addition to vertical stratification with constant strength N_v^2, this base flow includes an additional, horizontal, density gradient characterized by $N_h^2(t)$. The stability of this flow is then analyzed: When the vertical stratification is stabilizing, there is a simple harmonic motion of the horizontal stratification $N_h^2(t)$ and of the shear rate $S(t)$, but this flow is unstable to certain disturbances, which are amplified by a Floquet mechanism. This analysis may involve an additional Coriolis effect with Coriolis parameter f, so that governing dimensionless parameters are a modified Richardson number, $R=[S(0)^2+N_h^4(0)/N_v^2]^{1/2}$, and $f_v=f/N_v$, as well as the initial phase of the periodic shear rate. Parametric resonance between the inertia-gravity waves and the oscillating shear is demonstrated from the dispersion relation in the limit $R\rightarrow 0$. The parametric instability has connection with both baroclinic and elliptical flow instabilities, but can develop from a very different base flow.

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I. INTRODUCTION

An important class of solutions of the Euler equations can be found by superposing a "base" flow, which is linear in the space coordinates, and a "disturbance" flow which consists of plane waves with a time-dependent wave vector. Several studies were carried out in this context, with application to turbulence and hydrodynamic stability theory. Two-dimensional base flows have elliptical, linear, or hyperbolical streamlines. In addition to the hyperbolic instability simply recovered in the latter case (Batchelor and Proudman [1] and Lagnado *et al.* [2]), the case with elliptical streamlines yielded elegant analytical identification of the elliptical instability by Bayly [3] and Craik and Criminale [4], following the computational evidence by Pierrehumbert [5]. Even if the elliptical instability was known before (see the reviews by Cambon and Scott [6] and by Kerswell [7]), the later studies engendered new interest and refined approaches [8]. The same analysis was further extended to Euler equations with body forces, such as Coriolis and buoyancy forces [9,10], and with time-dependent strains [11]. The buoyant case allowed considering density as linear in space too, with application to the elliptical flow undergoing stable stratification, with [13] and without [12] system rotation.

The studies presented above gave access to several basic instabilities: Hyperbolical [2], elliptical [3,8], and barotropic [14], not to mention more specific parametric instabilities of gravity waves with periodic forcing [15]. In the latter case of buoyant flows with density gradients, another type of generic instability is the baroclinic instability; in contrast to previous studies [13,15], this instability can exist only in the presence of density gradient *in two directions* (not only aligned with the gravitational acceleration), as described by Pedlowsky [18] and Drazin and Reid [19]. In this context, Salhi and Cambon [14] proposed another angle of attack, with a basic flow combining vertical shear with constant rate S, Coriolis parameter f, and both vertical and horizontal density gradients.

The present study has strong analogies with the latter one, but the case with time-dependent shear rate offers other perspectives. In a different context, Craik and Allen [11] have investigated the class of time-periodic basic flows with spatially uniform strain rates by concentrating on sinusoidal strain rates. They noted that *these flows can be thought of as idealizations of local features within turbulent flows that display fluctuating rates of strain due to external excitation, either deliberately imposed at boundaries or caused by neighboring "eddies."* In these previous works, as also in [15], the physical meaning of these sinusoidal strain rates, that are admissible solutions, was not more clearly specified. The periodic shear rate, if *a priori* imposed, is no longer related to a real flow in our present study. On the other hand, only the buoyancy force with preexisting (constant rate) vertical stratification is responsible for the oscillatory motion of both the shear rate and the horizontal stratification rate, if an initial shear flow is present.

II. GOVERNING EQUATIONS

A. Euler equations

In the Boussinesq approximation, velocity and buoyancy fields $\mathbf{u}$ and b are described by the following Euler equations:

$$\nabla\cdot\mathbf{u}=0,\quad (\partial_t+\mathbf{u}\cdot\nabla)\mathbf{u}=-\nabla p+b\mathbf{n}, \tag{1}$$

$$(\partial_t+\mathbf{u}\cdot\nabla)b=0, \tag{2}$$

where p is the pressure divided by a fixed reference density ρ_0, $\mathbf{n}$ denotes an upward vertical unit vector, and b is the magnitude of the buoyancy force per unit mass, which is proportional to the departure of ambient density from refer-

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ence density.[1] Accordingly, the flow is assumed incompressible with variable density, but density variation is only accounted for when multiplied by the gravitational acceleration via the buoyancy term; diffusivity is neglected for both $\mathbf{u}$ and b.

Decomposing the flow into a basic state and a disturbance, as $\mathbf{u}=\mathbf{U}+\mathbf{u}'$, $p=P+p'$, and $b=B+b'$, the basic state $\mathbf{U}$, P, B ought to be governed by the equations above. This flow must be linear in space. The basic velocity field is chosen as a time-dependent shear flow in the (x_2,x_3) plane with uniform gradient

$$U_i(\mathbf{x},t) = S(t)x_2\delta_{i3}, \tag{3}$$

with a basic buoyancy term

$$B(\mathbf{x},t) = C_i(t)x_i \quad (i=1,2,3), \tag{4}$$

where the buoyancy gradient $\mathbf{C}=\boldsymbol{\nabla}B$ is not *a priori* specified.

B. Exact solutions for the base flow

Admissibility conditions are found by substituting the above relationship for $\mathbf{U}$, P, and B into Eqs. (1) and (2). It is easier to start with the curl of Eq. (1), which amounts to the equation for basic vorticity

$$(\partial_t + \mathbf{U}\cdot\boldsymbol{\nabla})\mathbf{W} = \mathbf{W}\cdot\boldsymbol{\nabla}\mathbf{U} + \boldsymbol{\nabla}\times(B\mathbf{n}), \tag{5}$$

with $\mathbf{W}=\boldsymbol{\nabla}\times\mathbf{U}$. First and second components yield

$$\frac{dS(t)}{dt} - C_2(t) = 0, \quad C_1(t) = 0, \tag{6}$$

whereas Eq. (4) implies the additional conditions

$$\frac{dC_2(t)}{dt} + S(t)C_3 = 0, \quad dC_3(t)/dt = 0. \tag{7}$$

(Brunt Waisala) vertical N_v and horizontal N_h buoyancy frequencies are classically defined as

$$N_v^2 = C_3 = \text{const}, \quad N_h^2(t) = C_2(t). \tag{8}$$

Two stratification gradients are therefore present.

C. Possible analogy with baroclinic instability

In the same way, the basic state considered by [14] involved a constant shear rate S_0 (i.e., $U_i=S_0x_3\delta_{i1}$), a constant vertical stratification N_v, and an additional Coriolis force with system vorticity $f\mathbf{n}$, f being the Coriolis parameter. The equivalent of the basic vorticity equation [first equation in Eq. (6)] was $0=S_0f-C_2(0)$, so that the tendency for the horizontal basic buoyancy gradient to generate streamwise vorticity was balanced by twisting the system vorticity f via the S_0f term. This effect is often called the geostrophic adjustment, or the thermal wind effect, in the geophysical community [19]. In the present case, the term $\partial_t\mathbf{W}=[dS(t)/dt]\mathbf{e}_1$ (vorticity derivative in the spanwise direction) plays a similar role in equilibrating the buoyancy term $\boldsymbol{\nabla}\times(B\mathbf{n})$, which corresponds to the horizontal buoyancy gradient.

The solution of the above differential equations for $S(t)$ and $C_2(t)$ are periodic for $N_v^2>0$ (i.e., stable vertical stratification), linear for $N_v=0$ (i.e., there is no vertical stratification), and exponential when $N_h^2<0$ (i.e., unstable vertical stratification). Even without initial basic shear $S(0)=S_0$, the simultaneous presence of vertical and horizontal stratifications (i.e., $N_h\neq 0$ and $N_v\neq 0$) may induce a nonzero shear gradient. In addition, if the shear rate ($S_0\neq 0$) and the vertical stratification ($N_v\neq 0$) are initially present, a horizontal density gradient is created. The "basic" state obtained at $N_v=0$ corresponds to a velocity field depending linearly on both time and space coordinate x_2 with a constant horizontal density gradient. The Coriolis force is no longer needed to trigger the baroclinic instability, but it is possible to reintroduce it, with system vorticity $f\mathbf{n}$, without changing the admissibility conditions, in order to generalize our study.

III. STABILITY ANALYSIS

A. The disturbance flow

Only considering the case of vertical stabilizing stratification $N_v^2>0$ with an additional Coriolis force with system vorticity $f\mathbf{n}$, from now on, basic flow solutions are

$$S(t) = A_0\cos(N_vt-\phi),$$

$$N_h^2(t) = -(A_0N_v)\sin(N_vt-\phi),$$

in which A_0 is the amplitude and ϕ is the initial phase, such that

$$A_0^2 = S^2(0) + N_h^4(0)/N_v^2, \quad \tan\phi = N_h^2(0)/[S(0)N_v],$$

so that $S(0)=A_0\cos\phi$ and $N_h^2(0)=A_0N_v\sin\phi$. When A_0 vanishes [i.e., $S(0)=0$ and $N_h(0)=0$], one recovers the case of a stable vertical stratification.

Equations for the *disturbance state* ($\mathbf{u}'$, p', b') are easily found. The equation for $\mathbf{u}'$, p' is not recalled here for the sake of brevity [14]; the equation for b' is

$$(\partial_t + \mathbf{u}'\cdot\boldsymbol{\nabla})b' + S(t)x_2\frac{\partial b'}{\partial x_3} = N_v^2u_3' + N_h^2(t)u_2'.$$

Their solutions are found in the form of single plane waves with a time-dependent wave vector

$$(u_i',p',b') = \mathrm{Re}\left\{(\hat{u}_i,\hat{p},\hat{b})\exp(\imath\mathbf{k}(t)\cdot\mathbf{x})\right\}. \tag{9}$$

The wave number advection equation (eikonal type) is $dk_i/dt+k_j(\partial U_j/\partial x_i)=0$, with solution

$$k_1 = K_1, \quad k_3 = K_3,$$

$$k_2(t) = K_2 - K_3\int_0^t S(t')dt' = K_2 - K_3R[\sin\phi + \sin(N_vt-\phi)], \tag{10}$$

where the capital letter denotes the initial value and $R=A_0/N_v$. It clearly appears that the wave number $\mathbf{k}(t)$ is time

[1]Different relationships between b, density, and temperature can be found in a liquid or a gas, which do not concern us here, since the above equations are the same in terms of b.

periodic, with period $2\pi/N_v$. In addition, the other time-dependent terms in the equations, like N_h^2, have the same period. Therefore, the Fourier coefficients $\hat{u}_i(t)$, $\hat{p}$, and $\hat{b}(t)$ of the inviscid and nondiffusive problem can be found from the solutions of a Floquet problem (e.g., Bayly [3] and Yakubovich and Starzhinskii [17]).

B. Floquet problem with a minimal number of components

The governing equations for the velocity and density components of this single-mode disturbance turn out to be linear, without approximation, as stressed by Craik [9], among other authors.

The linear system of ordinary differential equations (ODE) for the five Fourier coefficients ($\hat{u}_1$, $\hat{u}_2$, $\hat{u}_3$, $\hat{p}$, $\hat{b}$) is easily reduced to four components, removing the pressure term by accounting for the incompressibility constraint, or $\mathbf{k}(t)\cdot\hat{\mathbf{u}}=0$. In addition, the latter condition allows us to express the velocity amplitude in terms of two independent components only, denoted $u^{(1)}$ and $u^{(2)}$. All details can be found in [14]; the use of the Craya-Herring frame, ($\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$, $\mathbf{e}^{(3)}$) such that $\mathbf{e}^{(1)}=(-k_3/k_\perp,0,k_1/k_\perp)^T$ and $\mathbf{e}^{(3)}=\mathbf{k}/k$, is similar to the procedure by [16]. Here, $k_\perp=\sqrt{k_1^2+k_3^2}$ and T denotes transpose. For consistency, the third dependent variable $\hat{b}$ is scaled to have the same dimension as the velocity coefficients, or

$$u^{(3)}(t) = -(1/N_v)\hat{b}(t). \tag{11}$$

Finally, the solution of the initial-value linear system of equations with three components is written as

$$u^{(i)}(t) = g_{ij}(t)u^{(j)}(0), \tag{12}$$

with $(i,j=1,2,3)$, and the Floquet problem is then described by

$$\frac{d\mathbf{g}}{d\tau} = \mathbf{m}(\tau)\mathbf{g}, \quad \mathbf{g}(0) = I_3, \tag{13}$$

where I_3 is the unit matrix and $\mathbf{m}(t)$ is a 3×3 time-periodic matrix,

$$\mathbf{m} = \begin{pmatrix} 0 & f_v\dfrac{k_3}{k} + \dfrac{k_1}{k}\dfrac{S(t)}{N_v} & -\dfrac{k_1}{k_\perp} \\ -f_v\dfrac{k_3}{k} & \dfrac{k_2k_3}{k^2}\dfrac{S(t)}{N_v} & -\dfrac{k_2k_3}{k_\perp k} \\ \dfrac{k_1}{k_\perp} & \dfrac{k_2k_3}{k_\perp k} - \dfrac{k_\perp}{k}\dfrac{N_h^2(t)}{N_v^2} & 0 \end{pmatrix}, \tag{14}$$

where $\tau=N_vt$ is a dimensionless time and $f_v=f/N_v$ such that $1/f_v^2$ is a modified Burger number (see [19]).

C. Constant of motion

A preliminary numerical aproach has suggested that the system of ODE for the $u^{(i)}$, in three components, can be reduced to a second order system. A systematic analytical search for constants of motion followed, resulting in the equation

$$k(K_1u^{(2)} - k_\perp m_{32}u^{(1)} + k_\perp m_{12}u^{(3)}) = \text{const}. \tag{15}$$

As a consequence, one of the eigenvalues of $\mathbf{g}$ is one after one period. Since the product of the three eigenvalues, or Det$\mathbf{g}$, is equal to K/k [a classical result for shear flows resulting from $m_{ii}=(1/k)(dk/d\tau)$], this product is also equal to one at every period.

It follows that one of the Floquet multipliers is equal to one and the other two are either real with the form $\{\lambda,1/\lambda\}$, or are complex conjugates lying on the unit circle. Accordingly, the only quantity we need to know in order to compute the two relevant eigenvalues is the trace of the matrix $\mathbf{g}$ after one period. Instability, with a growth rate σ, is found for

$$|\Delta| = |g_{ii}(2\pi) - 1| > 2 \quad \text{with } \sigma = \frac{1}{2\pi}\ln\left(\frac{|\Delta| + \sqrt{\Delta^2-4}}{2}\right).$$

For the special case with $K_3=0$ (i.e., an infinite wavelength in the vertical direction—which is the streamwise component for the shear), the wave vector $\mathbf{k}$ is time independent, and one easily derives the solutions for g_{ij},

$$g_{11} = g_{33} = \cos\tau, \quad g_{12} = (k_1/k)R\tau\cos(\tau-\phi),$$

$$g_{31} = -g_{13} = \sin\tau, \quad g_{32} = (k_1/k)R\tau\sin(\tau-\phi),$$

and $g_{2i}=\delta_{2i}$.

When $K_3\neq 0$, Eq. (13) is integrated numerically over one period using a fourth-order Runge-Kutta scheme with nondimensional time step $10^{-4}\pi$ to find Δ. For the sake of brevity only the results related to the case where the initial wave vector $\mathbf{K}$ lies in the planes

$$K_2 = k_3R\sin\phi, \tag{16}$$

so that $k_2(t)=-RK_3\sin(\tau-\phi)$, are discussed. Equation (16) implies that $K_2=0$ when $\phi=0$ [i.e., there is no horizontal initial stratification, $N_h(0)=0$] or $R=0$ [i.e., $N_h=0$ and $S(0)=0$, only stable vertical stratification].

IV. RESULTS AND DISCUSSION

Numerical results are collected in Figs. 1 and 2, plotting the neutral curves which correspond to a constant (small) growth rate 10^{-5}, in terms of two parameters: The modified Richardson number $R=A_0/N_v$ (horizontal axis of coordinates) and the orientation $\cos\theta=K_1/K$ (vertical axis). θ is the angle between the initial wave vector $\mathbf{K}$ and the unit vector in the spanwise direction 1.

A. Case without system rotation

Results without Coriolis force ($f_v=0$) are shown on Fig. 1. Two regions of instability correspond to $\Delta>2$; they extend along the entire R range (and include top-right and top-left corners in Fig. 1). Those related to $\Delta<-2$ are narrower than the thickness of the lines shown (to detect them, a very fine mesh for θ was needed), they emanate from the points $\cos\theta=\pm 1/2$ at very small R (full lines with dots in the figure). For very weak shear rate ($R\ll 1$), the instability bands reach the $R=0$ axis at $\cos\theta^+=\pm 1$ ($\theta^+=0,\pi$) in the former

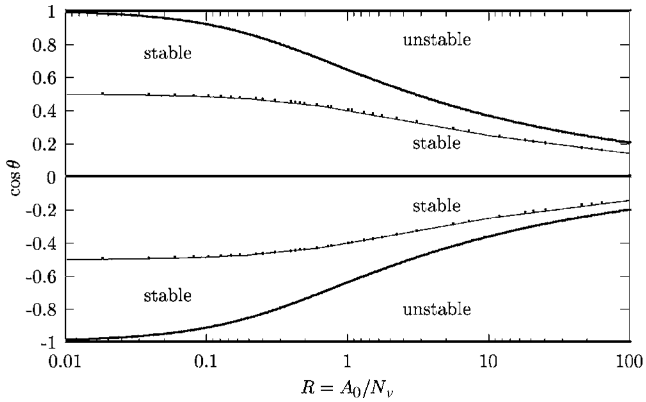

FIG. 1. Neutral curves for stability in the $(R, \cos\theta)$ plane. Those (full lines with dots) corresponding to $\Delta=-2$ reach the $R=0$ axis at $\cos\theta=\pm 1/2$.

case and at $\cos\theta^-=\pm 1/2$ ($\theta^-=\pi/3$ or $\theta^-=2\pi/3$) in the latter case (the superscript + or − for the angle refers to the sign of Δ). At the limiting points $R=0$ (shearless case), one has

$$g_{ii}(\tau) = 1 + 2\cos\sigma_0\tau, \tag{17}$$

where $\sigma_0=\cos\theta$ is nothing other than the unsigned nondimensional dispersion frequency of gravity waves. One recovers $\Delta=2$ at $\cos\theta^+=\pm 1$ and $\Delta=-2$ at $\cos\theta^-=\pm 1/2$. Accordingly, the four points (at $R=0$) from which the instability bands emanate give the relevant limit for a resonance between the gravity waves and the imposed (even vanishing) shear with additional horizontal stratification. This phenomenon is very similar to the resonance between the inertial wave and the imposed strain field in the elliptical instability [3,8,18]. The instability bands in the latter case emanated from the two points at vanishing strain rate (equivalent to $R=0$ here) for the nondimensional dispersion frequency $\pm 1/2$ of inertial waves. Here, the bandwidth of the instability for $\Delta>2$ increases as R increases (as did the elliptical flow bandwidth with increasing strain rate), and increases as the initial phase increases.

B. Additional system rotation

Reintroducing the Coriolis force, we now restrict our attention to the limiting case $R=0$ in order to determine the points located in the $R=0$ axis at which the resonance can occur (and from which the unstable regions can emanate). The classical dispersion relation of inertia-gravity waves,

$$\sigma_0^2 = f_v^2 + (1 - f_v^2)\cos^2\theta,$$

can be used (in a nondimensional form here, dividing the dispersion frequency by N_v), instead of the one for gravity waves alone ($\sigma_0^2=\cos^2\theta$), in Eq. (17). It follows that $\Delta=-2$ at $\cos^2\theta_1^-=[(1/4)-f_v^2]/(1-f_v^2)$ when $0\le f_v\le 1/2$ or $\cos^2\theta_{2p+1}^-=[(2p+1)^2/4-f_v^2]/(1-f_v^2)$ when $1<p+1/2\le f_v$, where p is an integer.

The numerical results obtained for $R=0.01$ corroborate the above analysis, for the first values $p=0$ and $p=1$, as shown by Fig. 2(a) displaying the regions of instability corresponding to $\Delta<-2$, regions which are delineated by $\cos\theta_p^-$ lines.

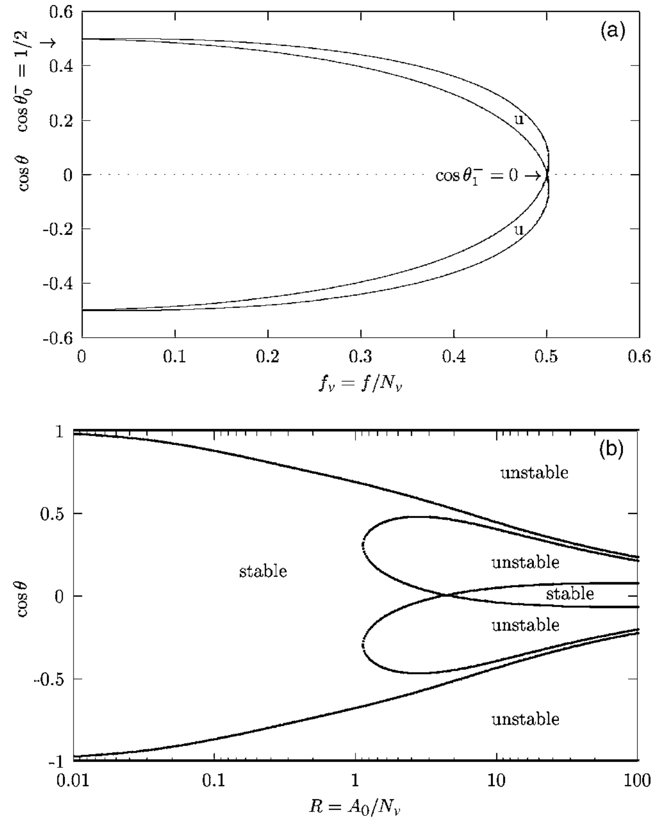

FIG. 2. Influence of f on the stability: (a) In the $(f_v, \cos\theta)$ plane for $R=0.01$. Neutral curves (full lines) corresponding to $\Delta=-2$ emanate from the $f_v=0$ axis at $\cos\theta_0^-=\pm 1/2$ and extend until $f_v=1/2$ (for which reconnection with the $\cos\theta_1^-=0$ case is recovered). (b) In the $(R, \cos\theta)$ plane for $f_v=3/4$, there are four unstable regions.

As for similar parametric instabilities [8,15], and considering their close analogy to Mathieu's forced pendulum, an alternative formulation of Eq. (13) would be a nonhomogeneous Hill's equation, $d^2V/d\tau^2+Z(\tau)V=h(\tau)$, where $Z(\tau)$ and $h(\tau)$ are periodic with period 2π. For weak shear flows (i.e., $R\ll 1$), perturbation techniques (e.g. [20]) can be used to determine instabilities of such an equation.

Figure 2(b) shows the instability bands in the $(R, \cos\theta)$ plane for $f_v=3/4$ with $\phi=0$. Only the instability bands corresponding to $\Delta<-2$ are strongly affected by the Coriolis force. One recovers the two regions of instability related to $\Delta>2$, emanating from ($\cos\theta=\pm 1$, $R\to 0$) that are not very affected by rotation, whereas the refined analysis of marginal stability at $\Delta=2$ based on the dispersion relation is not corroborated by numerics (this analysis, similar to the one for $\Delta=-2$, is therefore omitted here). In addition, four (new) instability bands appear for $R>1$, two for $\Delta>2$ and two for $\Delta<-2$. In contrast to the case without Coriolis force $f_v=0$, the latter two instability bands are not very thin.

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